Aharanov-Anandan geometric phase based precise quantum control and related topics

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in TQM symposium at Aalto Univ. (4 and 5 Nov. 2022)

Agenda

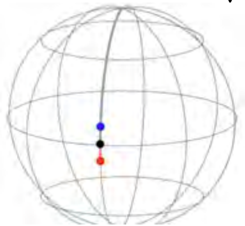
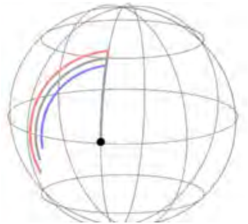
Who am I ?

- Graduated from Kyoto Univ. at 1989/05
- ROTA 1: 1988/11 ∼ 1991/8
- EP V, Uni. Bayreuth : 1991/9 ∼ 1995/10
- ROTA 1: 1995/11 ∼ 1997/3
- JRCAT: 1997/4 ∼ 2000/3
- Kindai Univ. : 2000/4 ∼
  - Quantum Computer related works
  - CREST (2017/10 ∼ 2023/3)

Precise Quantum Control with Erroneous Gates

Erroneous Gate and Composite Gate with them

$\sqrt{X}$gate

s  s 

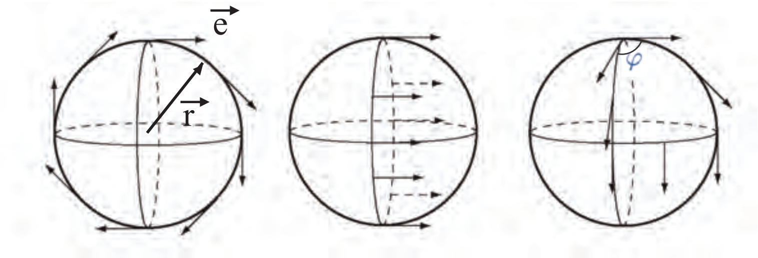
There are many composite pulses, such as

$90_x 180_y 90_x$, SCROFULOUS, W1 sequence.

We discuss:

- Why are they robust against errors ?
- Are there any common properties ?

Parallel Transport and Geometric Phase



The conditions of the parallel transport of $\vec{e}$ along a circuit C is

$$\vec{e} \cdot \vec{r} = 0$$

φ is equal to the solid angle enclosed by C.

Quantum Parallel Transport

The parallel transport law in Quantum Mechanics is

$$\Im(\langle \psi \,|d\psi\rangle) = 0, \text{ or } \Im(\langle \psi| \; \tfrac{d}{dt} \,|\psi\rangle) = 0.$$

Suppose $\Im(\langle \psi \,|d\psi\rangle) = 0$ is always satisfied and X (parameter of Hamiltonian) is taken round a circuit C. Then,

$$|\psi_{\text{final}}\rangle = \exp(i\gamma(C))\,|\psi_{\text{initial}}\rangle \,,$$

where $\gamma(C)$ is called as the quantum geometric phase since

$$\gamma(C) = \oint d\gamma = -\Im \oint_C \langle n\,|dn\rangle = -\Im \iint_{\partial S=C} \langle dn| \wedge |dn\rangle \,,$$

where $|\psi\rangle = |n(X)\rangle \exp(i\gamma)$.

Berry Phase

Schrödinger eq. $i\hbar\frac{d}{dt}|\Psi\rangle = H(X)|\Psi\rangle$ governs the dynamics of $|\Psi\rangle$. When X changes adiabatically, then

$$|\Psi\rangle \approx |\psi\rangle \exp\{-\frac{i}{\hbar}\int_0^t E_n(X(t'))dt'\}.$$

The adiabaticity leads $\Im(\langle\psi|\frac{d}{dt}|\psi\rangle) = 0$, and thus

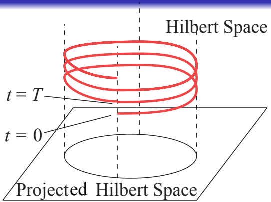
$$\langle\psi_{\mathrm{initial}}|\psi_{\mathrm{final}}\rangle = \exp(i\gamma(C)).$$

Finally, including the dynamic phase γ_{d},

$$\langle\Psi_{\mathrm{initial}}|\Psi_{\mathrm{final}}\rangle = \exp(i(\gamma_{\mathrm{d}} + \gamma(C))),$$

where $\gamma_{\mathrm{d}} = \exp\{-\frac{1}{\hbar}\int_0^T E_n(X(t'))dt'\}$ and T is a time for rounding the circuit C.

Aharonov-Anandan phase



We consider a cyclic motion in the projected Hilbert space that includes all quantum states but the phases are ignored, or

$$|\Psi(T)\rangle = e^{i\gamma}|\Psi(0)\rangle .$$

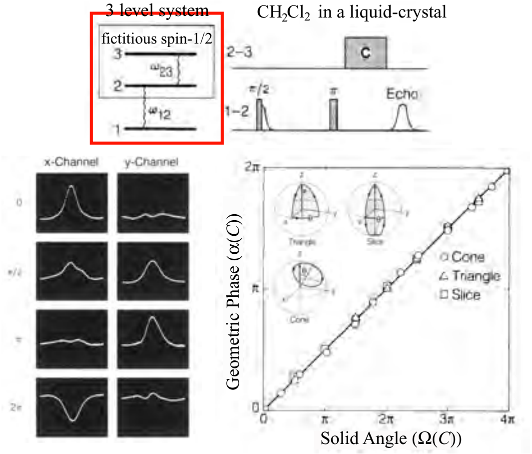
A dynamic phase $\gamma_{\rm d}$ and geometric phase $\gamma_{\rm g}$ are given as

$$\gamma_{\rm d} = -\int_0^T \langle\Psi| H |\Psi\rangle \, dt, \quad \gamma_{\rm g} = i\int_0^T \langle\Psi| \frac{d}{dt} |\Psi\rangle \, dt = \gamma - \gamma_{\rm d}.$$

The projected Hilbert space for a one-qubit can be mapped on to the Bloch sphere. When $H(\boldsymbol{m},\theta) = \theta\frac{\boldsymbol{m}\cdot\boldsymbol{\sigma}}{2}\frac{1}{T}$ and $|\Psi\rangle = |\boldsymbol{n}\rangle$,

$$\gamma_{\rm d} = -\int_0^T \langle\boldsymbol{n}|H(\boldsymbol{m},\theta)|\boldsymbol{n}\rangle dt = -\frac{\theta}{2}\boldsymbol{m}\cdot\boldsymbol{n}.$$

AA phase in NMR



D. Suter, etal, Phys. Rev. Lett. **60**, 1218 (1988).

Composite One-Qubit Gate

We consider an elementary gate $R(\boldsymbol{m}, \theta) = \exp\left(-i\theta \frac{\boldsymbol{m}\cdot\boldsymbol{\sigma}}{2}\right)$.
A real one qubit operation has an error ε, thus

$$\tilde{R}(\boldsymbol{m}, \theta) = R(\boldsymbol{m}, \theta(1+\varepsilon)) = R(\boldsymbol{m}, \theta) + O(\varepsilon).$$

We can construct a more reliable composite quantum gate,

$$\prod_{j=1}^{N} \tilde{R}(\boldsymbol{m}_j, \theta_j) = \prod_{j=1}^{N} \tilde{R}_j = R(\boldsymbol{n}_0, \theta) + O(\varepsilon^2).$$

We call this as a robust composite quantum gate against the ε .

Cyclic States and One-Qubit Gates

Any one-qubit gate U is given as

$$U = e^{i\gamma_+} |\boldsymbol{n}_+\rangle \langle \boldsymbol{n}_+| + e^{i\gamma_-} |\boldsymbol{n}_-\rangle \langle \boldsymbol{n}_-|,$$

where $|\boldsymbol{n}_\pm\rangle$ satisfies

$$\exists\, |\boldsymbol{n}_\pm\rangle \quad \text{s.t.} \quad U |\boldsymbol{n}_\pm\rangle = e^{i\gamma_\pm} |\boldsymbol{n}_\pm\rangle \quad \text{and} \quad \langle \boldsymbol{n}_+ | \boldsymbol{n}_-\rangle = 0,$$

and is called cyclic states. When $\gamma_\pm$ contains no dynamic phase, we call U as a geometric phase gate.
An elementary gate $R(\boldsymbol{m}, \theta) = \exp\left(-i\theta \frac{\boldsymbol{m}\cdot\boldsymbol{\sigma}}{2}\right)$ is also given as

$$R(\boldsymbol{m}, \theta) \quad = e^{-i\theta/2} |\boldsymbol{m}\rangle \langle \boldsymbol{m}| + e^{i\theta/2} |-\boldsymbol{m}\rangle \langle -\boldsymbol{m}|,$$

with cyclic states $|\pm\boldsymbol{m}\rangle$. But, it is a dynamic phase gate since $\gamma_{\mathrm{d}} = \mp\frac{\theta}{2}$ and $\gamma_{\mathrm{g}} = 0$.

Error of Composite Quantum Gate

$$\begin{aligned}
\prod_{j=1}^{N} \tilde{R}(\boldsymbol{m}_j, \theta_j) &= R(\boldsymbol{n}_0, \theta) + \sum_{j=1}^{N} R_N \dots R_j R(\boldsymbol{m}_j, \varepsilon\theta_j) \dots R_1 + O(\varepsilon^2) \\
&= R(\boldsymbol{n}_0, \theta) + \sum_{j=1}^{N} R_N \dots R_j \left(-i\varepsilon \frac{\theta_j}{2} \boldsymbol{m}_j \cdot \boldsymbol{\sigma} \right) \dots R_1 + O(\varepsilon^2) \\
&= R(\boldsymbol{n}_0, \theta) - i\varepsilon \sum_{j=1}^{N} R_N \dots R_j (H_j T_j) \dots R_1 + O(\varepsilon^2),
\end{aligned}$$

remember that $H_j T_j = \theta_j \boldsymbol{m} \cdot \boldsymbol{\sigma}/2$.

$\prod_{j=1}^{N} \tilde{R}(\boldsymbol{m}_j, \theta_j)$ becomes a robust composite quantum gate, if

$$\sum_{j=1}^{N} R_N \dots R_j (H_j T_j) \dots R_1 = 0.$$

Dynamic Phase and Robustness

The sum of dynamic phases during $\prod_{j=1}^{N} R_j$ starting from $|\boldsymbol{n}_0\rangle$ is given as

$$\sum_{j=1}^{N} \gamma_{\mathrm{d},j} = -\sum_{j=1}^{N} \langle \boldsymbol{n}_{j-1}| H_j T_j |\boldsymbol{n}_{j-1}\rangle ,$$

where $|\boldsymbol{n}_j\rangle = \prod_{k=1}^{j} R_k |\boldsymbol{n}_0\rangle$. On the other hand,

$$\langle \boldsymbol{n}_0| \sum_{j=1}^{N} R_N \dots R_j H_j T_j \dots R_1 |\boldsymbol{n}_0\rangle = e^{-i\theta/2} \sum_{j=1}^{N} \langle \boldsymbol{n}_{j-1}| H_j T_j |\boldsymbol{n}_{j-1}\rangle ,$$

since $\langle \boldsymbol{n}_0| R_N \dots R_j = e^{-i\theta/2} \langle \boldsymbol{n}_{j-1}|$.

A composite quantum gate is robust against a control field strength error.

↓

It is geometric.

Off-Resonance Error

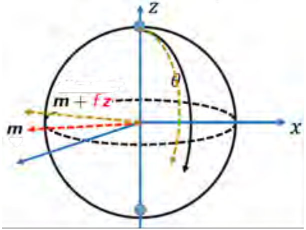
Two typical errors:

- Control field strength error $\longrightarrow$ AA phase based CQGs (called as Pulse Length Error)

$$\tilde{R}(\boldsymbol{m}, \theta) = R(\boldsymbol{m}, \theta(1+\varepsilon)) = R(\boldsymbol{m}, \theta) + O(\varepsilon)$$

- Off-Resonance Error $\longrightarrow$ Another type of CQGs

$$\tilde{R}(\boldsymbol{m}, \theta) = R(\boldsymbol{m} + f\boldsymbol{z}, \theta) = R(\boldsymbol{m}, \theta) + O(f)$$

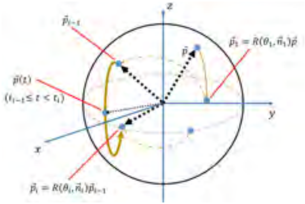


Geometric Property of ORE robust Composite QG

A CQG is ORE robust, iff its trajectory satisfies

$$\vec{z}^t \cdot \int_0^T \vec{p}(t)dt \quad = \quad 0,$$

for any starting point $\vec{p}(0)$ on the Bloch sphere.



When the angular velocity is constant,

$$\vec{z}^t \cdot \vec{M}_{\vec{p}} \quad = \quad 0.$$

A CQG is ORE robust iff the mass center $\vec{M}_{\vec{p}}$ of the errorless trajectory exists on the xy-plane.

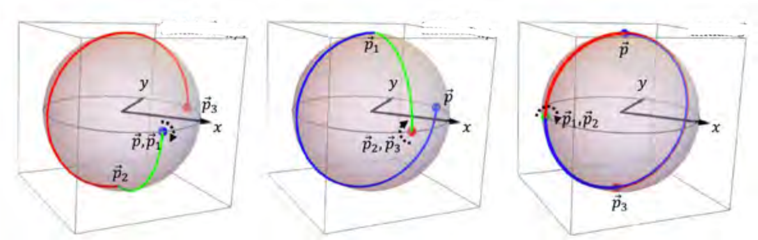
CORP^{2}SE

CORP^{2}SE

Compensation for Off Resonance with a Perpendicularly combined Pulse SEquence

as an ORE robust Composite Quantum Gate

$$
\begin{aligned}
& R(\boldsymbol{m}(0), \pi) \\
= \; & R(\boldsymbol{m}(3\pi/4), 3\pi/2) R(\boldsymbol{m}(\pi/4), \pi/2) R(\boldsymbol{m}(3\pi/4), 3\pi/2)
\end{aligned}
$$



initial: $\boldsymbol{m}(-\pi/4)$ $\quad\quad$ $\boldsymbol{m}(\pi/4)$ $\quad\quad$ $\boldsymbol{z}$

Conclusion

- Geometric properties of composite quantum gates are discussed.
  - JPSJ **80** (2011) 054002, Phil. Trans. R. Soc. **370** 2012, 4671.
  - Scientific Reports **12** (2022) 1.
- I have been working on composite quantum gates (11 papers) since 2009 with Bando, Ichikawa, Filgueiras, Goto, Güngördü, Kiya, Kukita, Nakahara, Nemoto, Ota, Shikano, and Suter.
- I will keep studying composite quantum gates.
  Application to
  - weakly anharmonic oscillators
  - quantum computer with globally manipulated qubits

Thank you for your attention.